

Available online at www.sciencedirect.com

J. Differential Equations 235 (2007) 74–84

**Journal of
Differential
Equations**

www.elsevier.com/locate/jde

Singular limit problem of the Camassa–Holm type equation

Seok Hwang

Department of Mathematics, LaGrange College, LaGrange, GA 30240, USA

Received 1 July 2005; revised 7 July 2006

Available online 9 January 2007

Abstract

We consider a shallow water equation of Camassa–Holm type, containing nonlinear dispersive effects as well as fourth order dissipative effects. We prove the strong convergence and establish the condition under which, as diffusion and dispersion parameters tend to zero, smooth solutions of the shallow water equation converge to the entropy solution of a scalar conservation law using methodology developed by Hwang and Tzavaras [S. Hwang, A.E. Tzavaras, Kinetic decomposition of approximate solutions to conservation laws: Applications to relaxation and diffusion–dispersion approximations, *Comm. Partial Differential Equations* 27 (2002) 1229–1254]. The proof relies on the kinetic formulation of conservation laws and the averaging lemma.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Shallow water equation; Conservation laws; Singular limit; Kinetic formulation; Averaging lemmas

1. Introduction

Recently, Coclite and Karlsen [6] showed the strong convergence of solutions $u^{\alpha, \beta, \gamma}$, as $\alpha, \beta, \gamma \rightarrow 0$, to the following scalar nonlinear partial differential equation

$$\partial_t u + \partial_x f(u) = \alpha \partial_{xx} u + 2\alpha \partial_x u \cdot \partial_{xx} u + \alpha u \cdot \partial_{xxx} u + \beta \partial_{xx} u - \gamma \partial_{xxxx} u, \quad (1.1)$$

where $f(u)$ is a smooth, genuinely nonlinear, and at most quadratically growing function. They proved that if $\alpha = O(\beta^4)$, $\gamma = O(\beta^5)$, then $u^{\alpha, \beta, \gamma}$ converges strongly to a limit function u that

E-mail address: shwang@lagrange.edu.

is a weak solution of the nonlinear conservation law

$$\partial_t u + \partial_x f(u) = 0. \quad (1.2)$$

Also, they proved that, under the stronger condition

$$\alpha = o(\beta^4), \quad \gamma = o(\beta^5), \quad (1.3)$$

the limit function u dissipates energy, that is, it satisfies the entropy inequality

$$\partial_t \left(\frac{u^2}{2} \right) + \partial_x q(u) \leq 0, \quad q'(u) = u f'(u) \quad \text{in } \mathcal{D}'.$$

However, it still remained an open problem if the limit function u dissipates all convex entropies, i.e., if u is a unique entropy solution. In this paper, we show that if the condition (1.3) is satisfied, then the limit function u is indeed a unique entropy solution. While Coclite and Karlsen applied the compensated compactness method in the L^p setting, we will use different methodology developed in [19] in this paper.

Recall that Camassa–Holm equation [3], which has received a considerable amount of attention in recent years, takes the form

$$\partial_t u + \kappa \partial_x u + 3u \cdot \partial_x u = \alpha \partial_{xx} u + 2\alpha \partial_x u \cdot \partial_{xx} u + \alpha u \cdot \partial_{xxx} u. \quad (1.4)$$

Note that (1.4) can be obtained by taking $\beta, \gamma = 0$ and $f(u) = \kappa u + \frac{3}{2}u^2$ in (1.1). The Camassa–Holm equation models the propagation of unidirectional shallow water waves on a flat bottom, and then $u(x, t)$ represents the fluid velocity at time t in the horizontal direction x [3,20]. Within this context, $\alpha > 0$ is a length scale (related to the shallowness) and $\kappa \geq 0$ is a constant that is proportional to the square root of water depth (see also Dai and Huo [13] for another interpretation related to a cylindrical compressible hyper-elastic rods).

The Camassa–Holm equation has many remarkable properties: it has a bi-Hamiltonian structure (and thus an infinite number of conservation laws) [3,16] and, as in the case of the KdV equation but not the BBM equation, it is completely integrable [1,3,11]. Moreover, when $\kappa = 0$ it has an infinite number of non-smooth solitary wave solutions called peakons, which interact like solitons. Although the KdV equation admits solitary waves that are solitons, it does not model wave breaking. On the other hand, the Camassa–Holm equation admits soliton solutions and at the same time allows for wave breaking.

The Cauchy problem of the Camassa–Holm equation has been well studied. Local well-posedness results are proved in [8,17,21,27]. It is also known that there exist global solutions for a certain class of initial data and solutions that blow up in finite time for a large class of initial data [7,8,10]. Existence and uniqueness results for global weak solutions of (1.4) have been proved in [9,12,14,15,30,31]. The generalized Camassa–Holm equation (1.1) with $\beta = 0$ was analyzed in [4], while the easier case when $\beta > 0$ is contained as a special case of a more general class of equations analyzed in [5].

We recall that the theory of compensated compactness (see [29]) says the compactness of a given family $\{u^\varepsilon\}$ of approximate solutions to scalar conservation laws bounded in some L^p -norm ($p > 1$) is determined by compactness of the entropy dissipation measure in the sense

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \quad \text{is precompact in } H_{\text{loc},x,t}^{-1}, \quad (1.5)$$

where η – q is an entropy–entropy flux pair with $q'(u) = f'(u)\eta'(u)$.

This has been proved in one-space dimension in both the L^∞ and L^p stability settings by Tartar [29] and Schonbek [28] (see [26] for a simplified proof using singular entropies). In [19], the authors show how the kinetic formulation compactness framework of Lions–Perthame–Tadmor [22] can be easily adapted to analyze the structure (1.5). More precisely, the entropy production is turned into a kinetic form using duality (see Section 3) and results to an approximate transport equation,

$$\partial_t \chi^\beta + f'(\xi) \cdot \partial_x \chi^\beta = \partial_x (\bar{g}^\beta + \partial_\xi g^\beta) + \partial_\xi k^\beta, \quad (1.6)$$

for the function $\chi^\beta = \mathbb{1}(u^\beta(x, t), \xi)$, where

$$\mathbb{1}(u, \xi) = \begin{cases} \mathbb{1}_{0 < \xi < u} & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -\mathbb{1}_{u < \xi < 0} & \text{if } u < 0, \end{cases} \quad (1.7)$$

is the usual Maxwellian associated with the kinetic formulation of scalar conservation laws, $\bar{g}^\beta, g^\beta \rightarrow 0$ in $L^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$ and k^β is uniformly bounded in measures. Convergence is then obtained via the averaging lemma in [25]. In the limit $\beta \rightarrow 0$, $\chi^\beta \rightarrow \mathbb{1}(u, \xi) =: \chi$ which satisfies

$$\partial_t \chi + f'(\xi) \cdot \partial_x \chi = \partial_\xi k \quad \text{in } D'_{x,t,\xi}, \quad (1.8)$$

with k a bounded measure. For the approximation (1.1), it turns out that if $\alpha = o(\beta^4)$, $\gamma = o(\beta^5)$, the bounded measure k is positive and u an entropy solution. By contrast, if $\alpha = O(\beta^4)$, $\gamma = O(\beta^5)$, the measure k might in general be nonpositive.

2. Main results

Consider the Camassa–Holm type equation

$$\begin{aligned} \partial_t u + \partial_x f(u) &= \alpha \partial_{xxt} u + 2\alpha \partial_x u \cdot \partial_{xx} u + \alpha u \cdot \partial_{xxx} u + \beta \partial_{xx} u - \gamma \partial_{xxxx} u, \quad x \in \mathbb{R}, \quad t \geq 0, \\ u(x, 0) &= u_0^{\alpha, \beta, \gamma}(x), \quad x \in \mathbb{R}. \end{aligned} \quad (2.1)$$

The objective is to show that solutions $u^{\alpha, \beta, \gamma}$ of the Camassa–Holm type equation (2.1) converge as $\alpha, \beta, \gamma \rightarrow 0$ towards a weak solution u of the scalar conservation law

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned} \quad (2.2)$$

We assume that the flux $f(u) : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 function satisfying

$$|f'(u)| \leq C_1 |u|, \quad |f(u)| \leq C_2 |u|^2, \quad u \in \mathbb{R}. \quad (2.3)$$

We also assume that the initial data u_0 satisfies

$$u_0 \in L^4(\mathbb{R}) \cap L^2(\mathbb{R}), \quad (2.4)$$

and the initial data $u_0^{\alpha,\beta,\gamma}$ satisfies

$$u_0^{\alpha,\beta,\gamma} \in H^2(\mathbb{R}), \quad u_0^{\alpha,\beta,\gamma} \rightarrow u_0 \quad \text{in } L^4(\mathbb{R}) \cap L^2(\mathbb{R}), \quad \text{as } \alpha, \beta, \gamma \rightarrow 0, \quad (2.5)$$

and

$$\begin{aligned} & \|u_0^{\alpha,\beta,\gamma}\|_{L^1(\mathbb{R})} + \|u_0^{\alpha,\beta,\gamma}\|_{L^2(\mathbb{R})} + \|u_0^{\alpha,\beta,\gamma}\|_{L^4(\mathbb{R})} \\ & + (\beta^4 + \beta\sqrt{\beta^2 + 1}) \|\partial_x u_0^{\alpha,\beta,\gamma}\|_{L^2(\mathbb{R})} + \beta^3 \sqrt{\beta^2 + 1} \|\partial_{xx} u_0^{\alpha,\beta,\gamma}\|_{L^2(\mathbb{R})} \leq C, \end{aligned} \quad (2.6)$$

for some constant $C > 0$ that is independent of α, β, γ .

The main result of this paper is the following:

Theorem 2.1. *Suppose that $f(u)$ satisfies (2.3) and the following non-degeneracy condition:*

$$\text{meas}\{\xi \in \mathbb{R} \mid f'(\xi) = s\} = 0, \quad \forall s \in \mathbb{R}. \quad (2.7)$$

- (i) *If $\alpha = O(\beta^4)$, $\gamma = O(\beta^5)$, then solutions u^β of (2.1) converge along a subsequence to a function u in $L^q_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$, $1 \leq q < 4$; the limiting u is a weak solution of (2.2).*
- (ii) *If $\alpha = o(\beta^4)$, $\gamma = o(\beta^5)$, then u is the unique Kruzhkov entropy solution of (2.2).*

3. Proof of the main results

In preparation, recall that η – q is an entropy–entropy flux pair if $q'(u) = f'(u)\eta'(u)$. Such pairs describe the nonlinear structure of (2.2) and are represented in terms of the kernel $\mathbb{1}(u, \xi)$ by the formulas

$$\begin{aligned} \eta(u) - \eta(0) &= \int_{\xi} \mathbb{1}(u, \xi) \eta'(\xi) d\xi, \\ q(u) - q(0) &= \int_{\xi} \mathbb{1}(u, \xi) f'(\xi) \eta'(\xi) d\xi. \end{aligned} \quad (3.1)$$

We begin with some estimates on smooth solutions $u^{\alpha,\beta,\gamma}$ of (2.1) (see [6] for a proof). Here we use the notation $u^\varepsilon \in_b X$ to denote sequences that are uniformly bounded in the norm of the Banach space X .

Lemma 3.1. *(See [6].) Assume the data $u_0^{\alpha,\beta,\gamma}$ satisfies the uniform bounds (2.4)–(2.6). Then, if $\alpha = O(\beta^4)$, $\gamma = O(\beta^5)$, we have*

$$u^{\alpha,\beta,\gamma}(x, t) \in_b L^\infty(\mathbb{R}^+; L^2(\mathbb{R})), \quad (3.2)$$

$$u^{\alpha,\beta,\gamma}(x, t) \in_b L^\infty(\mathbb{R}^+; L^4(\mathbb{R})), \quad (3.3)$$

$$\beta |\partial_x u^{\alpha,\beta,\gamma}(x, t)|^2 \in_b L^1(\mathbb{R} \times \mathbb{R}^+), \quad (3.4)$$

$$\alpha \beta |\partial_x u^{\alpha,\beta,\gamma}(x, t)|^4 \in_b L^1(\mathbb{R} \times \mathbb{R}^+), \quad (3.5)$$

$$\alpha \beta |\partial_{xx} u^{\alpha,\beta,\gamma}(x, t)|^2 \in_b L^1(\mathbb{R} \times \mathbb{R}^+), \quad (3.6)$$

$$\alpha\beta|\partial_{xt}u^{\alpha,\beta,\gamma}(x,t)|^2 \in_b L^1(\mathbb{R} \times \mathbb{R}^+), \quad (3.7)$$

$$\beta^4\gamma|\partial_{xxx}u^{\alpha,\beta,\gamma}(x,t)|^2 \in_b L^1(\mathbb{R} \times \mathbb{R}^+), \quad (3.8)$$

$$\beta|u^{\alpha,\beta,\gamma}(x,t)\partial_x u^{\alpha,\beta,\gamma}(x,t)|^2 \in_b L^1(\mathbb{R} \times \mathbb{R}^+), \quad (3.9)$$

$$\alpha\beta|u^{\alpha,\beta,\gamma}(x,t)\partial_{xx}u^{\alpha,\beta,\gamma}(x,t)|^2 \in_b L^1(\mathbb{R} \times \mathbb{R}^+). \quad (3.10)$$

Remark 3.2. Let $\mathbb{1}(u, \xi)$ be the entropy kernel. Since $u^\beta \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}))$ we have for K compact subset of $\mathbb{R} \times \mathbb{R}^+$

$$\int_K \left(\int_\xi |\mathbb{1}(u^\varepsilon, \xi)| d\xi \right)^2 dx dt = \int_K |u^\beta|^2 dx dt \leq C$$

and thus $\mathbb{1}(u^\beta, \xi) \in L^2_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+; L^1(\mathbb{R}))$.

We use the limiting case of the averaging lemma proved in Perthame–Souganidis [25], see also [24]:

Theorem 3.3. Let $\{f_n\}, \{g_{i,n}\}$ be two sequences of solutions to the transport equation

$$\partial_t f_n + a(\xi) \cdot \nabla_x f_n = \partial_t \partial_\xi^k g_{0,n} + \sum_{i=1}^d \partial_{x_i} \partial_\xi^k g_{i,n} \quad (3.11)$$

where $k \in \mathbb{N}$. Assume that $a(\xi) \in C^\infty(\mathbb{R})$ satisfies the non-degeneracy condition: for $R > 0$

$$\omega(\beta) = \sup_{\alpha \in \mathbb{R}, \omega \in S^{d-1}} \int_{\{|\xi| \leq R\}} \left(\left| \alpha + \frac{a(\xi) \cdot \omega}{\beta} \right|^2 + 1 \right)^{-1} d\xi \rightarrow 0, \quad \text{as } \beta \rightarrow 0. \quad (3.12)$$

If $\{f_n\}$ is bounded in $L^q(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, for some $1 < q < \infty$, and $\{g_{i,n}\}$ is precompact in $L^q(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$, then the average

$$\int_{\mathbb{R}} \psi(\xi) f_n(t, x, \xi) d\xi \quad \text{is precompact in } L^q(\mathbb{R}^d \times \mathbb{R}^+),$$

for any $\psi \in C_c^\infty(\mathbb{R})$.

Remark 3.4. 1. The non-degeneracy condition (3.12) is equivalent to for all $R > 0$

$$\text{meas}\{\xi \in B_R \mid \alpha + a(\xi) \cdot \omega = 0\} = 0, \quad \forall \alpha \in \mathbb{R}, \omega \in S^{d-1}, \quad (3.13)$$

where $B_R = \{|\xi| \leq R\}$. The condition (3.13) can be interpreted geometrically, and means that the curve $\xi \mapsto a(\xi)$ is not locally contained in any hyperplane $a(\xi) \cdot \omega + \alpha = 0$. Also, in 1-dimensional case ($d = 1$), when $f'(\xi) = a(\xi)$, it is easy to see that the condition (3.13) is equivalent to the condition (2.7).

2. An assumption on the behavior of $a(\xi)$ is necessary; there would be no improvement of regularity in the case $a(\xi) = \text{const}$ (see, for example, [2]).

3. By using cut-off functions, it is easy to show a variant of Theorem 3.3 stating that under the same hypotheses if $\{f_n\}$ is bounded in $L^q_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ and $\{g_{i,n}\}$ are precompact in $L^q_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$ then the averages are precompact in $L^q_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^+)$ for any $\psi \in C^\infty_c(\mathbb{R})$.

Proof of Theorem 2.1. Let $\alpha = O(\beta^4)$, $\gamma = O(\beta^5)$ and denote by $u^\beta = u^{\alpha,\beta,\gamma}$. We multiply (2.1) by $\eta'(u^\beta)$ and obtain

$$\begin{aligned} \partial_t \eta(u^\beta) + \partial_x q(u^\beta) &= \alpha \partial_x (\eta'(u^\beta) \partial_{xt} u^\beta) - \alpha \eta''(u^\beta) (\partial_x u^\beta) (\partial_{xt} u^\beta) \\ &\quad + \frac{\alpha}{2} \partial_x (\eta'(u^\beta) (\partial_x u^\beta)^2) - \frac{\alpha}{2} \eta''(u^\beta) (\partial_x u^\beta) (\partial_x u^\beta)^2 \\ &\quad + \alpha \partial_x (\eta'(u^\beta) u^\beta \partial_{xx} u^\beta) - \alpha \eta''(u^\beta) (\partial_x u^\beta) (u^\beta \partial_{xx} u^\beta) \\ &\quad + \beta \partial_x (\eta'(u^\beta) \partial_x u^\beta) - \beta \eta''(u^\beta) (\partial_x u^\beta)^2 \\ &\quad - \gamma \partial_x (\eta'(u^\beta) \partial_{xxx} u^\beta) + \gamma \eta''(u^\beta) (\partial_x u^\beta) (\partial_{xxx} u^\beta). \end{aligned} \quad (3.14)$$

Let $\varphi(x, t) \in C^\infty_c(\mathbb{R} \times \mathbb{R}^+)$ and let $\eta \in C^\infty_c(\mathbb{R})$ be viewed as a test function. By introducing the indicator function $\mathbb{1}(u^\beta, \xi)$, we have

$$\begin{aligned} & - \int_{x,t,\xi} (\mathbb{1}(u^\beta, \xi) \partial_t \varphi(x, t) + f'(\xi) \mathbb{1}(u^\beta, \xi) \partial_x \varphi(x, t)) \eta'(\xi) d\xi dx dt \\ &= - \int_{x,t} \left(\alpha \partial_{xt} u^\beta + \frac{\alpha}{2} (\partial_x u^\beta)^2 + \alpha u^\beta \partial_{xx} u^\beta + \beta \partial_x u^\beta - \gamma \partial_{xxx} u^\beta \right) \eta'(u^\beta) \partial_x \varphi(x, t) dx dt \\ &\quad - \int_{x,t} \eta''(u^\beta) \left(\alpha (\partial_x u^\beta) (\partial_{xt} u^\beta) + \frac{\alpha}{2} (\partial_x u^\beta) (\partial_x u^\beta)^2 + \alpha (\partial_x u^\beta) (u^\beta \partial_{xx} u^\beta) + \beta (\partial_x u^\beta)^2 \right. \\ &\quad \left. - \gamma (\partial_x u^\beta) (\partial_{xxx} u^\beta) \right) \varphi(x, t) dx dt, \end{aligned} \quad (3.15)$$

which is viewed as describing the action on tensor products $\varphi \otimes \eta'$.

We proceed to interpret (3.15) as an equation in $\mathcal{D}'_{x,t,\xi}$. Let

$$\begin{aligned} \chi^\beta &= \mathbb{1}(u^\beta, \xi), \\ H^\beta(x, t) &= \alpha \partial_{xt} u^\beta + \frac{\alpha}{2} (\partial_x u^\beta)^2 + \alpha u^\beta \partial_{xx} u^\beta + \beta \partial_x u^\beta - \gamma \partial_{xxx} u^\beta, \\ G^\beta(x, t) &= \alpha (\partial_x u^\beta) (\partial_{xt} u^\beta) + \frac{\alpha}{2} (\partial_x u^\beta) (\partial_x u^\beta)^2 + \alpha (\partial_x u^\beta) (u^\beta \partial_{xx} u^\beta) + \beta (\partial_x u^\beta)^2 \\ &\quad - \gamma (\partial_x u^\beta) (\partial_{xxx} u^\beta). \end{aligned}$$

Note that $H^\beta, G^\beta \in_b L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ from Lemma 3.1. We wish to define $\delta(u^\beta - \xi)G^\beta$ as a distribution in $\mathcal{D}'_{x,t,\xi}$ by its action on tensor products (same to $\delta(u^\beta - \xi)H^\beta$)

$$\langle \delta(u^\beta - \xi)G^\beta, \varphi \otimes \eta' \rangle = \int_{x,t} G^\beta(x, t) \varphi(x, t) \eta'(u^\beta(x, t)) dx dt. \quad (3.16)$$

This follows from the Schwartz kernel theorem (e.g. [18, Section 5.2]) as follows: Define the linear map

$$\mathcal{K}: C_c^\infty(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R} \times \mathbb{R}^+) \quad \text{by } \mathcal{K}\psi = G^\beta(x, t)\psi(u^\beta(x, t)).$$

If $\psi_j \rightarrow 0$ in $C_c^\infty(\mathbb{R})$ then $\mathcal{K}\psi_j \rightarrow 0$ in $\mathcal{D}'_{x,t}$. The kernel theorem implies that $\delta(u^\beta - \xi)G^\beta$ is well defined as a distribution in $\mathcal{D}'_{x,t,\xi}$ and acts on tensor products via (3.16). Moreover,

$$\langle \partial_\xi \delta(u^\beta - \xi)G^\beta, \varphi \otimes \eta' \rangle = - \int_{x,t} G^\beta(x, t) \varphi(x, t) \eta''(u^\beta(x, t)) dx dt. \quad (3.17)$$

Thus (3.15) is written as

$$\begin{aligned} & \langle \partial_t \chi^\beta + f'(\xi) \cdot \partial_x \chi^\beta, \eta'(\xi) \varphi(x, t) \rangle \\ &= \langle H^\beta, \eta'(\xi) \varphi(x, t) \rangle + \langle \partial_\xi (\delta(u^\beta - \xi)G^\beta), \eta'(\xi) \varphi(x, t) \rangle. \end{aligned}$$

Since the subspace generated by the direct sum test functions $\varphi \otimes \eta'$ is dense in $C_c^\infty(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$, the bracket (3.18) is extended to test functions $\theta(x, t, \xi) \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$. So, we have

$$\begin{aligned} \partial_t \chi^\beta + f'(\xi) \cdot \partial_x \chi^\beta &= \partial_x (H^\beta(x, t) \delta(u^\beta - \xi)) + \partial_\xi (G^\beta(x, t) \delta(u^\beta - \xi)) \\ &=: \partial_x \pi^\beta + \partial_\xi k^\beta \quad \text{in } \mathcal{D}'_{x,t,\xi}. \end{aligned} \quad (3.18)$$

We estimate first the terms π^β : Let $\theta(x, t, \xi) \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$. Using Lemma 3.1, we see that for $\alpha = O(\beta^4)$, $\gamma = O(\beta^5)$,

$$\begin{aligned} & |\langle H^\beta \delta(u^\beta - \xi), \theta(x, t, \xi) \rangle| \\ &= \left| \int_{x,t} \left(\alpha \partial_{xt} u^\beta + \frac{\alpha}{2} (\partial_x u^\beta)^2 + \alpha u^\beta \partial_{xx} u^\beta + \beta \partial_x u^\beta - \gamma \partial_{xxx} u^\beta \right) \theta(x, t, u^\beta(x, t)) dx dt \right| \\ &\leq \left\| \alpha |\partial_{xt} u^\beta| + \frac{\alpha}{2} |\partial_x u^\beta|^2 + \alpha |u^\beta \partial_{xx} u^\beta| + \beta |\partial_x u^\beta| + \gamma |\partial_{xxx} u^\beta| \right\|_{L^2_{x,t}} \cdot \|\theta(x, t, u^\beta)\|_{L^2_{x,t}} \\ &\leq C\beta^{1/2} (\|\beta^{5/2} \partial_{xt} u^\beta\|_{L^2_{x,t}} + \|\beta^{5/2} (\partial_x u^\beta)^2\|_{L^2_{x,t}} + \|\beta^{5/2} u^\beta \partial_{xx} u^\beta\|_{L^2_{x,t}} \\ &\quad + \|\beta^{1/2} \partial_x u^\beta\|_{L^2_{x,t}} + \|\beta^{9/2} \partial_{xxx} u^\beta\|_{L^2_{x,t}}) \|\theta\|_{L^2_{x,t}(H^1_\xi)} \\ &\leq C\beta^{1/2} \|\theta\|_{L^2_{x,t}(H^1_\xi)}. \end{aligned}$$

Here we used the following:

$$\begin{aligned} \int_{x,t} \theta^2(x, t, u^\beta) dx dt &= \int_{x,t} \int_{-\infty}^{u^\beta(x,t)} 2\theta\theta_\xi d\xi dx dt \\ &\leq 2 \int_{x,t} \left(\int_{-\infty}^{u^\beta} \theta^2 d\xi \right)^{\frac{1}{2}} \left(\int_{-\infty}^{u^\beta} (\partial_\xi \theta)^2 d\xi \right)^{\frac{1}{2}} dx dt \leq \|\theta\|_{L^2_{x,t}(H^1_\xi)}^2, \end{aligned} \quad (3.19)$$

and by (3.2),

$$\int_{x,t} (\partial_x u^\beta)^4 dx dt \leq C \int_t \left(\int_x (u^\beta)^2 dx \right) \left(\int_x (\partial_{xx} u^\beta)^2 dx \right) dt \leq C \int_{x,t} (\partial_{xx} u^\beta)^2 dx dt,$$

so by (3.6),

$$\alpha\beta \int_{x,t} (\partial_x u^\beta)^4 dx dt \leq C.$$

This shows that $\pi^\beta \rightarrow 0$ in $L^2_{x,t}(H^{-1}_\xi)$ as $\beta \rightarrow 0$, or in other words

$$\pi^\beta = \bar{g}^\beta + \partial_\xi g^\beta \quad \text{with } \bar{g}^\beta, g^\beta \rightarrow 0 \text{ in } L^2_{x,t,\xi}.$$

Next, consider the term $k^\beta = G^\beta \delta(u^\beta - \xi)$. Observe that

$$\begin{aligned} |G^\beta - \beta|u^\beta_x|^2| &\leq \alpha|u^\beta_x||u^\beta_{xt}| + \frac{\alpha}{2}|u^\beta_x||u^\beta_x|^2 + \alpha|u^\beta_x||u^\beta u^\beta_{xx}| + \gamma|u^\beta_x||u^\beta_{xxx}| \\ &\leq \frac{\alpha}{2\beta}|u^\beta_x|^2 + \frac{\alpha\beta}{2}|u^\beta_{xt}|^2 + \frac{\alpha}{4\beta}|u^\beta_x|^2 + \frac{\alpha\beta}{4}|u^\beta_x|^4 \\ &\quad + \frac{\alpha}{2\beta}|u^\beta_x|^2 + \frac{\alpha\beta}{2}|u^\beta u^\beta_{xx}|^2 + \frac{\gamma}{2\beta^4}|u^\beta_x|^2 + \frac{\gamma\beta^4}{2}|u^\beta_{xxx}|^2. \end{aligned} \quad (3.20)$$

If $\alpha = O(\beta^4)$ and $\gamma = O(\beta^5)$, the estimates in Lemma 3.1 imply $G^\beta \in_b L^1(\mathbb{R} \times \mathbb{R}^+)$. Thus

$$\begin{aligned} |(k^\beta, \theta)| &= |(\delta(u^\beta - \xi)G^\beta, \theta)| \\ &\leq \sup_{x,t,\xi} |\theta(x, t, \xi)| \cdot \|G^\beta\|_{L^1(\mathbb{R} \times \mathbb{R}^+)} \leq C\|\theta\|_{C^0}. \end{aligned}$$

In summary, the function $\chi^\beta = \mathbb{1}(u^\beta, \xi)$ satisfies the transport equation

$$\partial_t \chi^\beta + f'(\xi) \cdot \partial_x \chi^\beta = \partial_x (\bar{g}^\beta + \partial_\xi g^\beta) + \partial_\xi k^\beta \quad \text{in } D'_{x,t,\xi}, \quad (3.21)$$

where $\bar{g}^\beta, g^\beta \rightarrow 0$ in $L^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$ while k^β is bounded in measures $\mathcal{M}(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$ and precompact in $W_{\text{loc}}^{-1,p}(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$, for $1 \leq p < \frac{3}{2}$. By the averaging lemma (Theorem 3.3)

$$\int_{\xi} \mathbb{1}(u^\beta, \xi) \psi(\xi) d\xi \quad \text{is precompact in } L_{\text{loc}}^p, \quad 1 < p < \frac{3}{2},$$

for $\psi(\xi) \in C_c^\infty(\mathbb{R})$.

Let R be a large positive number and consider $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi = 1$ on $(-R, R)$ and $0 \leq \psi \leq 1$. Then

$$\begin{aligned} \left| u^\beta - \int_{\mathbb{R}} \mathbb{1}(u^\beta, \xi) \psi(\xi) d\xi \right| &= \left| \int_{\mathbb{R}} \mathbb{1}(u^\beta, \xi) (1 - \psi(\xi)) d\xi \right| \\ &\leq \int_R^\infty |\mathbb{1}(u^\beta, \xi)| d\xi + \int_{-\infty}^{-R} |\mathbb{1}(u^\beta, \xi)| d\xi = (u^\beta - R)^+ + (u^\beta + R)^-. \end{aligned}$$

Moreover,

$$\begin{aligned} \int (u^\beta - R)^+ + (u^\beta + R)^- dx dt &\leq \int_{|u^\beta| > R} |u^\beta| dx dt \\ &\leq \frac{1}{R} \int_0^T \int |u^\beta|^2 dx dt \leq \frac{C}{R}. \end{aligned}$$

We conclude that $\{u^\beta\}$ is Cauchy in $L_{\text{loc},x,t}^1$. Since $u^\beta \in_b L^\infty(L^4 \cap L^2)$, it follows that (along subsequences) $u^\beta \rightarrow u$ in L_{loc}^p , $p < 4$, and almost everywhere and that $u \in L^\infty(L^4 \cap L^2)$.

To pass to the limit in (3.21), note that

$$\begin{aligned} \chi^\beta &= \mathbb{1}(u^\beta, \xi) \rightarrow \chi = \mathbb{1}(u, \xi) \quad \text{a.e. and in } L_{\text{loc},x,t}^p(L_\xi^p), \quad 1 \leq p < 4, \\ k^\beta &= G^\beta \delta(u^\beta - \xi) \rightharpoonup k \quad \text{weak-}\star \text{ in } \mathcal{M}_{x,t,\xi}, \end{aligned} \quad (3.22)$$

and thus χ satisfies

$$\partial_t \chi + f'(\xi) \cdot \partial_x \chi = \partial_\xi k \quad \text{in } D'_{x,t,\xi}. \quad (3.23)$$

For $\alpha = O(\beta^4)$ and $\gamma = O(\beta^5)$, the bounded measure k may, in general, be nonpositive. By contrast, for $\alpha = o(\beta^4)$ and $\gamma = o(\beta^5)$, the function $\chi = \mathbb{1}(u, \xi)$ satisfies the kinetic formulation of Lions–Perthame–Tadmor

$$\partial_t \chi + f'(\xi) \cdot \partial_x \chi = \partial_\xi m$$

with m a positive, bounded measure, and thus u is the unique entropy solution of (2.2) (see [23]). To see that, let m denote the weak- \star limit:

$$(\beta |\partial_x u^\beta|^2) \delta(u^\beta - \xi) \rightharpoonup m \quad \text{weak-}\star \text{ in measures.}$$

By Lemma 3.1 and (3.20), we have for $\alpha = o(\beta^4)$ and $\gamma = o(\beta^5)$,

$$|G^\beta - \beta |\partial_x u^\beta|^2| \rightarrow 0 \quad \text{in } L^1_{x,t},$$

and thus $k = m \geq 0$. \square

References

- [1] R. Beals, D.H. Sattinger, J. Szmigielski, Acoustic scattering and the extended Korteweg–de Vries hierarchy, *Adv. Math.* 140 (2) (1998) 190–206.
- [2] F. Bouchut, Introduction to the mathematical theory of kinetic equations, in: F. Bouchut, F. Golse, M. Pulvirenti (Eds.), *Kinetic Equations and Asymptotic Theories*, in: Ser. Appl. Math., vol. 4, Elsevier, 2000.
- [3] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* 71 (11) (1993) 1661–1664.
- [4] G.M. Coclite, H. Holden, K.H. Karlsen, Global weak solutions to a generalized hyper-elastic rod wave equation, *SIAM J. Math. Anal.* 37 (4) (2006) 1044–1069.
- [5] G.M. Coclite, H. Holden, K.H. Karlsen, Wellposedness for a parabolic–elliptic system, *Discrete Contin. Dyn. Syst.* 13 (3) (2005) 659–682.
- [6] G.M. Coclite, K.H. Karlsen, A singular limit problem for conservation laws related to the Camassa–Holm shallow water equation, *Comm. Partial Differential Equations* 31 (2006) 1253–1272.
- [7] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: A geometric approach, *Ann. Inst. Fourier (Grenoble)* 50 (2) (2000) 321–362.
- [8] A. Constantin, J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (4) 26 (2) (1998) 303–328.
- [9] A. Constantin, J. Escher, Global weak solutions for a shallow water equation, *Indiana Univ. Math. J.* 47 (4) (1998) 1527–1545.
- [10] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.* 181 (2) (1998) 229–243.
- [11] A. Constantin, H.P. McKean, A shallow water equation on the circle, *Comm. Pure Appl. Math.* 52 (8) (1999) 949–982.
- [12] A. Constantin, L. Molinet, Global weak solutions for a shallow water equation, *Comm. Math. Phys.* 211 (1) (2000) 45–61.
- [13] H.H. Dai, Y. Huo, Solitary shock waves and other traveling waves in a general compressible hyperelastic rod, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 456 (1994) (2000) 331–363.
- [14] R. Danchin, A few remarks on the Camassa–Holm equation, *Differential Integral Equations* 14 (8) (2001) 953–988.
- [15] R. Danchin, A note on well-posedness for Camassa–Holm equation, *J. Differential Equations* 192 (2) (2003) 429–444.
- [16] B. Fuchssteiner, A.S. Fokas, Symplectic structures, their Backlund transformations and hereditary symmetries, *Phys. D* 4 (1) (1981/1982) 47–66.
- [17] A.A. Himonas, A.S. Misiulek, The Cauchy problem for an integrable shallow water equation, *Differential Integral Equations* 14 (8) (2001) 953–988.
- [18] L. Hormander, *The Analysis of Linear Partial Differential Operators*, Springer, New York, 1990.
- [19] S. Hwang, A.E. Tzavaras, Kinetic decomposition of approximate solutions to conservation laws: Applications to relaxation and diffusion–dispersion approximations, *Comm. Partial Differential Equations* 27 (2002) 1229–1254.
- [20] R.S. Johnson, Korteweg–de Vries and related models for water waves, *J. Fluid Mech.* 455 (2002) 63–82.
- [21] Y.A. Li, P.J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Differential Equations* 162 (1) (2000) 27–63.
- [22] P.L. Lions, B. Perthame, E. Tadmor, A kinetic formulation of scalar multidimensional conservation laws, *J. Amer. Math. Soc.* 7 (1994) 169–191.

- [23] B. Perthame, Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure, *J. Math. Pures Appl.* 77 (1998) 1055–1064.
- [24] B. Perthame, Lecture notes on kinetic formulation of conservation laws, in: L. Hsiao, Z. Xin (Eds.), *Some Current Topics on Nonlinear Conservation Laws*, in: *Stud. Adv. Math.*, vol. 5, Amer. Math. Soc./International Press, 2000, pp. 111–140.
- [25] B. Perthame, P.E. Souganidis, A limiting case for velocity averaging, *Ann. Sci. Ecole Norm. Sup.* 31 (1998) 591–598.
- [26] B. Perthame, A.E. Tzavaras, Kinetic formulation for systems of two conservation laws and elastodynamics, *Arch. Ration. Mech. Anal.* 155 (2000) 1–48.
- [27] G. Rodriguez-Blanco, On the Cauchy problem for the Camassa–Holm equation, *Nonlinear Anal.* 46 (3) (2001) 309–327.
- [28] M.E. Schonbek, Convergence of solutions to nonlinear dispersive equations, *Comm. Partial Differential Equations* 7 (1982) 959–1000.
- [29] L. Tartar, Compensated compactness and applications to partial differential equations, in: R.J. Knops (Ed.), *Nonlinear Analysis and Mechanics, Heriot–Watt Symposium*, in: *Pitman Res. Notes in Math.*, vol. 4, Pitman, Boston, 1979, pp. 136–192.
- [30] Z. Xin, P. Zhang, On the weak solutions to a shallow water equation, *Comm. Pure Appl. Math.* 53 (11) (2000) 1411–1433.
- [31] Z. Xin, P. Zhang, On the uniqueness and large time behavior of the weak solutions to a shallow water equation, *Comm. Partial Differential Equations* 27 (9–10) (2002) 1815–1844.